

On the gauge symmetries of the spinning particle

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Abstract

We reconsider the gauge symmetries of the spinning particle by a direct examination of the Lagrangian using a systematic procedure based on the Noether identities. It proves possible to find a set of local Bosonic and Fermionic gauge transformations that have a simple gauge group structure, which is a true Lie algebra, both for the massless and massive case. This new Fermionic gauge transformation of the “position” and “spin” variables in the action decouples from that of the “einbein” and “gravitino”.

It is also possible to redefine the fields so that this simple algebra of commutators of the gauge transformations can be derived directly starting from the Lagrangian written in these new variables.

We discuss a possible extension of our analysis of this simple model to more complicated cases.

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I. INTRODUCTION

For four decades, supersymmetry has been studied intensively. The local version of this symmetry, supergravity, is most easily realized by the spinning particle [1–3]; this is supergravity theory in 0+1 dimensions.

In the original presentation of the action for the spinning particle, a particular set of local Bosonic and Fermionic gauge transformations was given [1, 2]; their form appears to be motivated by the supersymmetric and diffeomorphism gauge transformations present in the supergravity action in 3+1 dimensions [4]. However, as was noted in [2], these gauge transformations do not have a gauge group structure in which the structure functions are field independent.

We wish in this paper to point out that this deficiency can be overcome by altering the form of the gauge transformations in a simple way. This is systematically done by direct derivation starting from the Lagrangian. A general form of an arbitrary gauge transformation can be derived from differential identities (DIs), which are linear combinations of Euler-Lagrange derivatives (ELDs) of the action; this method can be applied to any action with a known gauge transformation to search for a form of the local gauge transformations that simplifies the gauge group properties.

This general expression for a gauge transformation obtained from a DI of the action for the spinning particle can also be used to find a reparametrization of the fields so that the Fermionic gauge transformation decouples the “position” and “spin” fields from the “einbein” and “gravitino” fields.

We note that the gauge symmetry structure of the spinning particle action can also be studied using the canonical structure of the action; a generator of both Bosonic and Fermionic gauge transformations that have a simple gauge group structure can be derived from first class constraints [5]. The same procedure can be applied for the superparticle action [6].

II. THE SPINNING PARTICLE

We start by examining the general case of a particle action

$$S = \int d\tau L(q_i(\tau), \dot{q}_i(\tau)) \quad (1)$$

(where $\dot{q}_i(\tau) \equiv \frac{\partial q_i}{\partial \tau}$) and considering a variation of each of the fields $q_i(\tau)$ so that

$$\delta S = \int d\tau \sum_i \delta q_i(\tau) E_{q_i}. \quad (2)$$

In eq. (2), the Euler-Lagrange derivatives (ELDs) E_{q_i} are given by

$$E_{q_i} = \frac{\delta L}{\delta q_i} - \partial \frac{\delta L}{\delta \dot{q}_i} \quad (3)$$

(where $\partial \equiv \frac{\partial}{\partial \tau}$). If this were to vanish for arbitrary δq_i , then we have the fields q_i satisfying the equation of motion $E_{q_i} = 0$. However, if the form of δq_i is such that $\delta S = 0$ for arbitrary $q_i(\tau)$, then we have a local gauge symmetry of the action S . According to the Noether theorem [7, 8], any gauge symmetry $q_i \rightarrow q_i + \delta q_i$ of S satisfies the equation

$$\sum_i \delta q_i E_{q_i} \equiv 0, \quad (4)$$

which leads to the differential identity I from

$$I\alpha = \sum_i \delta q_i(\tau) E_{q_i}, \quad (5)$$

where α is a gauge parameter.

The Lagrangian we are particularly concerned with is that of the spinning particle

$$L = \frac{1}{2} \left[e^{-1} \dot{\phi}^\mu \dot{\phi}_\mu - i \psi^\mu \dot{\psi}_\mu - i e^{-1} \chi \psi^\mu \dot{\phi}_\mu \right], \quad (6)$$

where $\phi^\mu(\tau)$ and $e(\tau)$ are two Bosonic fields and $\psi^\mu(\tau)$ and $\chi(\tau)$ are two Fermionic fields, or two pairs of superpartners (e, χ) and (ϕ_μ, ψ^μ) . The index μ in ϕ^μ and ψ^μ is raised and lowered by the Minkowski metric tensor.

In refs. [1, 2], the Bosonic gauge invariance present in this action is given by the “diffeomorphism” transformation

$$\delta_f e = \partial(fe), \quad \delta_f \chi = \partial(f\chi), \quad \delta_f \phi^\mu = f\partial(\phi^\mu), \quad \delta_f \psi^\mu = f\partial(\psi^\mu), \quad (7)$$

while the Fermionic or supersymmetry transformation was chosen by the authors of [1, 2]

as

$$\delta_\alpha \phi^\mu = i\alpha \psi^\mu, \quad \delta_\alpha e = i\alpha \chi, \quad \delta_\alpha \chi = 2\dot{\alpha}, \quad \delta_\alpha \psi^\mu = \alpha e^{-1} \left(\dot{\phi}^\mu - \frac{i}{2} \chi \psi^\mu \right) \quad (8)$$

with $f(\tau)$ and $\alpha(\tau)$ being Bosonic and Fermionic gauge parameters, respectively.

Under the transformations of (7) and (8) we find

$$\delta_f L = \partial(fL), \quad (9)$$

$$\delta_\alpha L = \partial \left(\frac{i}{2e} \alpha \psi^\mu \dot{\phi}_\mu \right), \quad (10)$$

respectively.

The group property of the Bosonic transformation of (7) can be combined into one equation

$$[\delta_{f_1}, \delta_{f_2}](\phi^\mu, \psi^\mu, e, \chi) = \delta_{\tilde{f}}(\phi^\mu, \psi^\mu, e, \chi), \quad (11)$$

where

$$\tilde{f} = f_2 \dot{f}_1 - f_1 \dot{f}_2 \quad (12)$$

and so the structure functions of this transformation are field independent. The same is valid also for the commutator of Bosonic and Fermionic transformations

$$[\delta_f, \delta_\alpha](\phi^\mu, \psi^\mu, e, \chi) = \delta_{\tilde{\alpha}}(\phi^\mu, \psi^\mu, e, \chi) \quad (13)$$

with the parameter

$$\tilde{\alpha} = -f\dot{\alpha}. \quad (14)$$

However, when commuting two Fermionic gauge transformations we find from eqs. (7) and (8)

$$[\delta_{\alpha_1}, \delta_{\alpha_2}](\phi^\mu, \psi^\mu, e, \chi) = \delta_{\tilde{f}}(\phi^\mu, \psi^\mu, e, \chi) + \delta_{\tilde{\alpha}}(\phi^\mu, \psi^\mu, e, \chi), \quad (15)$$

where

$$\bar{f} = -\frac{2i\alpha_1\alpha_2}{e} \quad \text{and} \quad \bar{\alpha} = -\frac{1}{2}\bar{f}\chi. \quad (16)$$

We thus see that the transformations of eqs. (7) and (8) do not possess a simple group property, as was noticed by the authors of [2], because of the explicit dependence of the gauge parameters in eq. (16) on the fields.

We will now exploit eq. (4), using the Noether identities, to find a form of gauge transformations that has a simpler gauge group. This method was outlined in [9, 10]. We begin from the ELDs of the Lagrangian (6)

$$E_e = \frac{\delta L}{\delta e} = -\frac{1}{2e^2} \left(\dot{\phi}^\mu \dot{\phi}_\mu - i\chi\psi^\mu \dot{\phi}_\mu \right), \quad (17)$$

$$E_\chi = \frac{\delta L}{\delta \chi} = -\frac{i}{2e} \psi^\mu \dot{\phi}_\mu, \quad (18)$$

$$E_{\phi^\mu} = \frac{\delta L}{\delta \phi^\mu} = -\partial \left(e^{-1} \dot{\phi}_\mu - \frac{i}{2e} \chi \psi_\mu \right), \quad (19)$$

$$E_{\psi^\mu} = \frac{\delta L}{\delta \psi^\mu} = -i\dot{\psi}_\mu + \frac{i}{2e} \chi \dot{\phi}_\mu, \quad (20)$$

and so by eqs. (7) and (8) respectively we have the Bosonic DI

$$I = -e\partial E_e - \chi\partial E_\chi + \partial(\phi^\mu) E_{\phi_\mu} + \partial(\psi^\mu) E_{\psi_\mu} \equiv 0 \quad (21)$$

and the Fermionic DI

$$\Psi = -2\partial E_\chi + i\chi E_e + i\psi^\mu E_{\phi_\mu} + \left(\dot{\phi}^\mu - \frac{i}{2}\chi\psi^\mu \right) e^{-1} E_{\psi_\mu} \equiv 0. \quad (22)$$

We now use the fact that any linear combination of these two DIs yields a DI; the number of linearly independent DIs cannot be changed [7, 8]. The simplest modification is to multiply eq. (22) by a function $h(e)$, if we want to preserve its tensorial and Fermionic nature; the resulting DI corresponds to the gauge transformations

$$\delta_\alpha \chi = \partial(2\alpha h(e)), \quad \delta_\alpha e = i\alpha \chi h(e), \quad (23)$$

$$\delta_\alpha \phi^\mu = i\alpha \psi^\mu h(e), \quad \delta_\alpha \psi^\mu = \alpha e^{-1} \left(\dot{\phi}^\mu - \frac{i}{2}\chi\psi^\mu \right) h(e), \quad (24)$$

which gives the commutator

$$[\delta_{\alpha_1}, \delta_{\alpha_2}] \chi = \partial \left(-4i\alpha_1\alpha_2 \frac{dh(e)}{de} h(e) \chi \right). \quad (25)$$

All field dependence in this commutator disappears if we take

$$h(e) = \sqrt{e}, \quad (26)$$

so that the Fermionic gauge transformation becomes

$$\delta_\alpha \phi^\mu = i\sqrt{e}\alpha\psi^\mu, \quad \delta_\alpha e = i\sqrt{e}\alpha\chi, \quad \delta_\alpha \chi = 2\partial(\sqrt{e}\alpha), \quad \delta_\alpha \psi^\mu = \alpha \frac{1}{\sqrt{e}} \left(\dot{\phi}^\mu - \frac{i}{2}\chi\psi^\mu \right). \quad (27)$$

The commutator of two new Fermionic gauge transformations is now becomes

$$[\delta_{\alpha_1}, \delta_{\alpha_2}] (\phi^\mu, \psi^\mu, e, \chi) = \delta_{\bar{f}} (\phi^\mu, \psi^\mu, e, \chi) \quad (28)$$

with

$$\bar{f} = -2i\alpha_1\alpha_2, \quad (29)$$

while together the Fermionic and Bosonic gauge transformations result in

$$[\delta_f, \delta_\alpha] (\phi^\mu, \psi^\mu, e, \chi) = \delta_{\tilde{\alpha}} (\phi^\mu, \psi^\mu, e, \chi) \quad (30)$$

with

$$\tilde{\alpha} = f\dot{\alpha} - \frac{1}{2}\alpha\dot{f}. \quad (31)$$

With the gauge transformations of eqs. (11) and (27) we see that we have a gauge algebra whose structure functions are independent of fields; this simple algebra automatically satisfies the Jacobi identities.

One can supplement the Lagrangian of eq. (6) with a “mass”, or “cosmological” term

$$L_5 = \frac{1}{2} \left(m^2 e + i\psi_5 \dot{\psi}_5 - im\psi_5 \chi \right). \quad (32)$$

The Lagrangian L_5 of eq. (32) is invariant under the gauge transformations of eqs. (7) and (8) provided we also have [1]

$$\delta_f \psi_5 = f \dot{\psi}_5 \quad (33)$$

and

$$\delta_\alpha \psi_5 = m\alpha, \quad (34)$$

which result in

$$\delta_f L_5 = \partial(f L_5), \quad \delta_\alpha L_5 = \partial\left(\alpha \frac{i}{2} m \psi_5\right); \quad (35)$$

and the commutator of two Fermionic transformations is very simple:

$$[\delta_{\alpha_1}, \delta_{\alpha_2}] \psi_5 = 0. \quad (36)$$

However, the transformation of eq. (34) has been supplemented by an extra piece given in [2, 3] so that

$$\delta_\alpha \psi_5 = m\alpha + \frac{i}{me} \alpha \psi_5 \left(\dot{\psi}_5 - \frac{1}{2} m \chi \right) \quad (37)$$

in order that the gauge algebra of eq. (15) is satisfied.

The ELD associated with ψ_5 is

$$E_{\psi_5} = i \dot{\psi}_5 - \frac{i}{2} m \chi; \quad (38)$$

when this is combined with the Fermionic gauge transformation of eq. (37) we end up with the DI

$$\Psi \Rightarrow \Psi + \left[m + \frac{i}{me} \psi_5 \left(\dot{\psi}_5 - \frac{1}{2} m \chi \right) \right] E_{\psi_5} \equiv 0, \quad (39)$$

where Ψ is given by eq. (22) with new “einbein” and “gravitino” ELDs are

$$E_e \Rightarrow E_e + \frac{1}{2} m^2, \quad (40)$$

$$E_\chi \Rightarrow E_\chi + \frac{i}{2} m \psi_5. \quad (41)$$

We would like to note that, despite of consistency with the gauge algebra of eq. (15), the gauge transformation (37) with the extra piece is not legitimate because it introduces

a term in the DI proportional to the square of ELD E_{ψ_5} , as eq. (39) can be written in the form

$$\Psi + mE_{\psi_5} + \frac{1}{me}\psi_5 (E_{\psi_5})^2 \equiv 0. \quad (42)$$

It contradicts the definition of a DI as being a linear combination of ELDs.

We now make use of the function $h(e)$ of eq. (26) which modifies the DI so that we have the Fermionic gauge transformation

$$\delta_\alpha \psi_5 = \sqrt{e}m\alpha + \frac{i}{m\sqrt{e}}\alpha\psi_5 \left(\dot{\psi}_5 - \frac{1}{2}m\chi \right). \quad (43)$$

This is consistent with the gauge algebra given by eqs. (28-31).

One can also make an invertible change of variables in the original action without destroying the gauge algebra. For example, in ref. [1] a rescaling of Fermionic variables

$$\psi^\mu = \frac{1}{\sqrt{e}}\tilde{\psi}^\mu, \quad \chi = \frac{1}{\sqrt{e}}\tilde{\chi}, \quad (44)$$

$$\psi_5 = \sqrt{e}\tilde{\psi}_5 \quad (45)$$

leads to the actions of eqs. (6, 32) being replaced by

$$\tilde{L} = \frac{1}{2} \left[e^{-1}\dot{\phi}^\mu\dot{\phi}_\mu - ie^{-1}\tilde{\psi}^\mu\partial\left(\tilde{\psi}_\mu\right) - ie^{-2}\tilde{\chi}\tilde{\psi}^\mu\dot{\phi}_\mu \right] \quad (46)$$

and

$$\tilde{L}_5 = \frac{1}{2} \left(m^2e + ie\tilde{\psi}_5\partial\left(\tilde{\psi}_5\right) - im\tilde{\psi}_5\tilde{\chi} \right). \quad (47)$$

One can easily obtain the DI in terms of these new variables. The resulting gauge transformations for these new variables are

$$\delta_f \tilde{\chi} = \partial(f\tilde{\chi}) + \frac{1}{2}\dot{f}\tilde{\chi}, \quad \delta_f \tilde{\psi}^\mu = f\partial\left(\tilde{\psi}^\mu\right) + \frac{1}{2}\dot{f}\tilde{\psi}^\mu, \quad \delta_f \tilde{\psi}_5 = f\partial\left(\tilde{\psi}_5\right) - \frac{1}{2}\dot{f}\tilde{\psi}_5 \quad (48)$$

for the Bosonic case (the gauge transformations $\delta_f e$ and $\delta_f \phi^\mu$ remains the same as in (7)), and

$$\delta_\alpha e = i\alpha\tilde{\chi}, \quad \delta_\alpha\tilde{\chi} = 2\dot{\alpha}e + \alpha\dot{e}, \quad (49)$$

$$\delta_\alpha\phi^\mu = i\alpha\tilde{\psi}^\mu, \quad \delta_\alpha\tilde{\psi}^\mu = \alpha\dot{\phi}^\mu, \quad (50)$$

$$\delta_\alpha\tilde{\psi}_5 = m\alpha + \frac{i}{m}\alpha\tilde{\psi}_5\partial\left(\tilde{\psi}_5\right) \quad (51)$$

for the Fermionic case. The Fermionic transformations of eq. (49, 50, 51) are much simpler than those of (8, 43). In addition, the transformations of ϕ^μ and $\tilde{\psi}^\mu$ (the “position” and “spin” fields) decouple from those of e and $\tilde{\chi}$ (the “einbein” and “gravitino” fields), as well as $\tilde{\psi}_5$ transforms separately from other fields.

Despite this new form of the gauge transformations, we retain the simple gauge algebra of eqs. (11, 12, 28-31) in which all structure functions are field independent:

$$[\delta_{f_1}, \delta_{f_2}] \text{ field} = \delta_{\tilde{f}} \text{ field}, \quad \text{with} \quad \tilde{f} = f_2\dot{f}_1 - f_1\dot{f}_2, \quad (52)$$

$$[\delta_{\alpha_1}, \delta_{\alpha_2}] \text{ field} = \delta_{\tilde{f}} \text{ field}, \quad \text{with} \quad \tilde{f} = -2i\alpha_1\alpha_2, \quad (53)$$

$$[\delta_f, \delta_\alpha] \text{ field} = \delta_{\tilde{\alpha}} \text{ field}, \quad \text{with} \quad \tilde{\alpha} = f\dot{\alpha} - \frac{1}{2}\alpha\dot{f}. \quad (54)$$

The Jacobi identities automatically hold for such gauge transformations.

A particularly simple form of the gauge transformations for the fields which almost trivializes the calculations of the gauge algebra can be obtained by replacing e by g and $\tilde{\chi}$ by χ'' where

$$e = \exp(g), \quad (55)$$

$$\chi'' = \exp(-g)\tilde{\chi}. \quad (56)$$

We now find the Bosonic transformation

$$\delta_f g = \dot{f} + f\dot{g}, \quad (57)$$

$$\delta_f \chi'' = f \dot{\chi}'' + \frac{1}{2} \dot{f} \chi'' \quad (58)$$

and the Fermionic one

$$\delta_\alpha \tilde{\psi}^\mu = \alpha \dot{\phi}^\mu, \quad \delta_\alpha \phi^\mu = i\alpha \tilde{\psi}^\mu, \quad (59)$$

$$\delta_\alpha g = i\alpha \chi'', \quad \delta_\alpha \chi'' = 2\dot{\alpha} + \alpha \dot{g}. \quad (60)$$

This last parametrization (55, 56) has especially simple transformations that makes calculation of commutators of two supersymmetry transformations almost trivial.

III. DISCUSSION

By working directly with the DIs obtained from the action for the spinning particle, we have derived a set of Bosonic (B) and Fermionic (F) gauge transformations that have a simple gauge algebra of the form

$$[B, B] = B, \quad [F, F] = B, \quad [F, B] = F. \quad (61)$$

In this algebra, all structure functions are field independent and the Jacobi identity is satisfied. This is an improvement over the original set of gauge transformations appearing in refs. [1, 2]. Note, if we are seeking for gauge transformations of Bosonic and Fermionic fields that form a Lie algebra with field independent structure functions, then the only form possible is that of eq. (61).

The actual form of the gauge transformations has been simplified through a field redefinition while retaining the simple algebra of (61) for the gauge transformations.

We would like to investigate more complicated models that have a local Fermionic symmetry to see if similar simplifications can be effected. An $O(N)$ generalization of the spinning particle, the spinning string and supergravity in $D \geq 3$ dimensions should all be examined with this objective in mind.

A general problem would be to establish the relationship between the DI of eq. (4) that is satisfied by any gauge transformation and the gauge generator obtained from the first class constraints arising from the canonical structure of the theory [11, 12].

The gauge generator derived from the first class constraints can be used to determine a gauge invariance of a theory (even one previously unsuspected, as in the case of the first order Einstein-Hilbert action in two dimensions [13]). However, it has not as yet proven possible to determine directly from the Lagrangian and its ELDs all independent DIs of the form (4) and consequently, all gauge symmetries, though once one has a gauge transformation, alternate gauge transformations can easily be found by using this DI.

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- [1] L. Brink, S. Deser, B. Zumino, P. Di Vecchia, P. Howe, Phys. Lett. B 64 (1976) 435-438
 - [2] L. Brink, P. Di Vecchia, P. Howe, Nucl. Phys. B 118 (1977) 76-94
 - [3] Erratum, Phys. Lett. B 68 (1977) 488
 - [4] P. van Nieuwenhuizen, Phys. Rep. 68 (1981) 189
 - [5] D.G.C. McKeon, Can. J. Phys. 90 (2012) 701-705
 - [6] D.G.C. McKeon, arXiv:1209.4909 [hep-th]
 - [7] Noether, E.: Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math.-phys. Klasse, 235 (1918)
 - [8] Noether, E. (M.A. Tavel's English translation): arXiv:physics/0503066
 - [9] N. Kiriushcheva and S. V. Kuzmin, Gen. Rel. Grav. 42 (2010) 2613-2631
 - [10] N. Kiriushcheva, P. G. Komorowski, and S. V. Kuzmin, arXiv:1112.5637 [hep-th]
 - [11] L. Castellani, Symmetries in Constrained Hamiltonian Systems, Ann. Phys. **143** (1982) 357-371
 - [12] M. Henneaux, C. Teitelboim and J. Zanelli, Nucl. Phys. B 332 (1990) 169
 - [13] N. Kiriushcheva, S. V. Kuzmin and D.G.C. McKeon, Mod. Phys. Lett. A 20 (2005) 1895